XV. Mathematical Symmetry in Vergil's *Aeneid*

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This article is a brief summary of some amazing discoveries which I have made in recent years concerning the structure of Vergil's *Aeneid*. During the academic year 1957–58 I was engaged upon a general book on *Vergil as the Poet of Augustan Rome*,[1] and in April, 1958, my analyses of the individual books (for a chapter on the structure of the *Aeneid*) revealed unexpectedly the basic symmetry of the poem. The symmetry is mathematical: each book reveals, in small units as well as in the main divisions, the famous numerical ratio known variously as the Golden Section, the Divine Proportion, or the Golden Mean ratio.

As a result of this discovery, my original work was laid aside for the time being, and I have now completed a new book, neither planned nor anticipated, with the title: *Structural Patterns and Proportions in Vergil's Aeneid*. The subject is a technical one, but it has the excitement of a journey into new and uncharted territory. If the presence of exact or approximate Golden Mean ratios everywhere in the *Aeneid* seems improbable or even fantastic, I can assure the reader that it seemed so likewise to me when I first discovered their existence. Each step in my investigations led to new and equally surprising results. In this resumé I shall first describe how my earlier work led me to the mathematical proportions; then I shall explain what they are, *how* and *why* (if possible) Vergil employed them to the extent that he did, and finally—most important of all—I shall attempt to show the significance of the proportions for our understanding of many problems concerning the text of the poem.

I

The *Aeneid* of Vergil is the great national epic of ancient Rome; it portrays the journey of Aeneas and the Trojans from Troy to

[1] Freedom from academic duties in 1957–58 was made possible by a leave of absence granted by Princeton University and a Fellowship awarded by the John Simon Guggenheim Memorial Foundation.

Italy and their trials and victories after they reach their "promised land"; it gives the archaeology and topography of early Latium and Rome, and by means of prophecy and foreshadowing it presents the outstanding events of Roman history and the achievements of Augustus in Vergil's own day. The miracle of the poem is Vergil's ability to treat three different topics simultaneously—the legendary narrative of Aeneas, themes and personages of Roman history, and the praise of Augustus who has brought a new era of peace to the Roman world. The epic rises far above the patriotic and historical level in the poet's dramatic treatment of character and event and in his introduction of loftier themes of philosophy and religion; it is an epic not only of Rome but of human life as well. Vergil's superb poetic power, as seen in imagery, sound effects (such as alliteration and assonance), and complex metrical patterns, contributes to the greatness and splendor of the poem. The *Aeneid* is one of the most consciously planned and carefully constructed poems of world literature.

In a work of such magnitude, with so many threads firmly and harmoniously interwoven, it should occasion no surprise that one or more basic designs of symmetry and variety, of parallelism and contrast, underlie the composition of the poem, both as a whole and in its separate parts. All readers of the *Aeneid* are conscious of these qualities to a degree, but the extent to which structural pattern and architectonic design dominate the epic has not been realized fully. Moreover, these structural features are not merely an adornment for their own sake but are devised to emphasize and make more significant the meaning of the poem; structure and content go hand in hand, and Vergil's artistry in combining the two is an additional proof of his supreme achievement as an epic poet.

The *Aeneid* reveals a conscious attention to various structural devices: alternation, parallelism by means of similarities and contrasts, concentric or framework patterns, tripartite divisions; these appear both in the epic as a whole and in the individual books. This in itself is not surprising, for earlier poetry, both Greek and Roman, had been composed with a similar devotion to structural design. Throughout the *Iliad* of Homer there exists a most elaborate correspondence of parts, in which, as Whitman says, "episodes, and even whole books, balance each other through similarity or opposition."[2] The fact that many or most in the

[2] C. H. Whitman, *Homer and the Heroic Tradition* (Cambridge [Mass.] 1958) 258.

Homeric audience would not be conscious of this elaborate design is no argument against its existence. A great poet, or artist, or musician, always puts more into a work than is ordinarily realized, and this is even truer of Vergil than of Homer.

In the *Aeneid* as a whole we find three definite structural patterns:

1. The alternation of the books, those with even numbers being of a more serious and tragic nature than those with odd numbers, which are lighter and serve to relieve the tension. The famous books which stand out in the reader's memory are even-numbered: II, the fall of Troy; IV, the tragedy of Dido; VI, the trip to the underworld; VIII, the visit to early Rome; X, the great battle, with the deaths of Pallas, Lausus, and Mezentius; and XII, the final conflict and the death of Turnus. Vergil has stressed the significance of these books by means of the alternating rhythm.

2. The second pattern is the parallelism, by similarity and contrast, between the books in each half, I and VII, II and VIII, III and IX, etc. I have discussed this elsewhere,[3] and therefore shall list only a few examples. The similarities between I and VII are very numerous; in each the Trojans arrive in a strange land and are welcomed after a speech by Ilioneus; in each Juno stirs up trouble with divine or infernal assistance—in I, the storm at sea, and in VII, the war in Latium. Book II is the fall of Troy and VIII the birth of Rome; at the end of II Aeneas carries on his shoulders from Troy his father—a symbol of the past; at the end of VIII he raises to his shoulder the shield portraying scenes of Roman history—symbolic of the future. Book IV is the tragedy of Love, and X is the tragedy of War. One of the best parallels I have noted only recently: at the end of VI Anchises recommends *clementia* and *iustitia* in the famous verse (853):

parcere subiectis et debellare superbos;

at the end of XII when the wounded Turnus appeals for mercy, Aeneas is about to show *clementia* and spare him, when he sees the swordbelt of Pallas. In X Vergil had referred to Turnus as *superbus* both before and after the killing of Pallas (445, 514), and

[3] G. E. Duckworth, "The Architecture of the *Aeneid*," *AJP* 75 (1954) 1–15.

so now, at the very end of the poem, Aeneas cannot yield to clemency; *iustitia* demands that Turnus die.[4]

3. Vergil combines with the alternation of the books and their division into two corresponding halves a third and most important architectonic device—a tripartite division of the epic into three groups of four books each.[5] The *Aeneid* is the story of Aeneas, but it is also the story of the destiny of Rome under Augustus. This latter provides much of the central core of the poem (v–viii) and concludes with the victories and triumphs of Augustus as described on the shield at the end of viii. The *Aeneid* is thus a trilogy with the first four books, the tragedy of Dido, and the last four books, the tragedy of Turnus, enclosing in a framework pattern the central portion, where long Homeric episodes (games, trip to underworld, catalogue, description of a shield) are reworked and transformed for the glorification of Rome and its history, the portrayal of ancient Italy, and the praise of Augustus and the new Golden Age. This division of the poem into three parts is undoubtedly a deliberate attempt on Vergil's part to avoid too sharp a break into an "Odyssey" of wanderings and an "Iliad" of battles, and it enables him to emphasize the story of

[4] Among other similarities and contrasts not listed in my 1954 article (above, note 3) the following may be mentioned:

i. Trojans *laeti* (35), and vii. Trojans and Aeneas *laeti* (36, 130, 147, 288);
i and vii. Juno laments her lack of power;
i. Pictures of Trojan past, and vii. Statues of Latin past;
i. Closing of gates of war (in Jupiter's prophecy), and vii. Opening of gates of war;
ii. Luxury of Priam's palace, and viii. Simplicity of Evander's home;
ii and viii. Venus as goddess appears to Aeneas;
ii. Gods against Troy, and viii. Gods for Rome (at Actium);
iii. Apollo helps Aeneas, and ix. Apollo advises Ascanius;
iii. Astyanax-Ascanius equation (489 ff.), and ix. Euryalus-Ascanius equation (297 ff.);
iv. Jupiter intervenes, and x. Jupiter refuses to intervene;
iv. Aeneas sheds tears, but fate prevails (449), and x. Hercules sheds tears, but fate prevails (464 f.) [On the *lacrimae* of iv. 449, as the tears of Aeneas, see A. S. Pease, *Publi Vergili Maronis Aeneidos Liber Quartus* (Cambridge [Mass.] 1935) 367 f.; V. Pöschl, *Die Dichtkunst Virgils* (Innsbrück 1950) 76 ff.];
v. Cavalry display (*ludus Troiae*), and xi. Cavalry battle;
v. Juno sends down Iris, and xi. Juno sends down Opis;
v. Palinurus killed by minor deity (Somnus), and xi. Arruns killed by minor deity (Opis);
vi. Meeting with Dido concludes Dido-story (iv), and xii. Avenging of Pallas concludes Pallas-story (x).

[5] For details, see G. E. Duckworth, "The *Aeneid* as a Trilogy," *TAPA* 88 (1957) 1–10.

Rome and Augustus in the very center of the epic[6]; we have here an excellent illustration of the manner in which structure and content are combined.

In the individual books likewise, Vergil introduces effective contrasts by means of alternation; e.g. in the contests in v where the first (boat race) and the third (boxing match) are longer and contain more details of characterization than the second (foot race) and the fourth (archery match).[7] The poet's use of alternation is especially striking in the catalogue of Latin warriors (vii.641–807); the leaders alternate between major figures (Mezentius and Lausus, Messapus, Turnus and Camilla) and those of less significance, the latter being arranged in three groups of three each[8]; also, there is an interesting geographical variety, with the warriors from local areas enclosing pairs from the north (Messapus and Clausus), the south (Halaesus and Oebalus), and the east (Ufens and Umbro); i.e. we have two types of alternation in the same passage.

A second design is that of a balanced symmetry around an important central passage or focal point; this is called a framework or concentric or recessed panel pattern. Mendell has pointed out several examples in various books of the *Aeneid*,[9] but he does not cite vi.56–123, where we find an almost perfect balance of passages about the words of the Sibyl (83–97)—an important speech which arouses suspense concerning the events of vii–xii:

[6] The temple to honor Octavian which Vergil describes in *Georg.* iii. 13 ff. is usually believed to refer to an historical epic in which he intended to praise the deeds of Octavian; cf. 16:

in medio mihi Caesar erit templumque tenebit.

It is significant that Vergil achieved the same result in his mythological epic about Aeneas and the Trojans. When we view the *Aeneid* as a trilogy, it becomes apparent that his desire to have Caesar *in medio* has been fulfilled. Augustus is at the very heart and center of the poem (vi.788–807), and the central portion (v–viii) concludes with the scenes on the shield describing his victory at Actium and his triumphs at Rome (viii.675–728). It should be noted also that on the shield itself Augustus occupies the central position; cf. *in medio* (viii.675).

[7] I do not include among the contests the *ludus Troiae* (545–603) which is a separate spectacle not previously announced (cf. 64–70); see below, note 12.

[8] See E. A. Hahn, "Vergil's Catalogue of the Latin Forces: A Reply to Professor Brotherton," *TAPA* 63 (1932) lxii f.

[9] C. W. Mendell, "The Influence of the Epyllion on the *Aeneid*," *YClS* 12 (1951) 222 ff. For similar patterns in Catullus 64, the longer poems of the *Appendix Vergiliana*, the *Eclogues*, and the *Georgics*, see L. Richardson, Jr., *Poetical Theory in Republican Rome* (New Haven 1944).

56–76 Speech of Aeneas
77–82 Description of the Sibyl
83–97 Speech of the Sibyl
98–102 Description of the Sibyl
103–123 Speech of Aeneas

This symmetrical framing of the Sibyl's speech underlines its importance for the later action of the poem.[10] *Aeneid* IV as a whole may be viewed as a series of loosely corresponding episodes about a focal point.[11] The speeches of Dido and Anna at the end balance those at the beginning, and Mercury appears to Aeneas twice and urges him to depart. In the very center we have the following sequence:

279–304 Narrative. Preparations for departure
305–30 Dido's speech
331–33a Aeneas' emotions
333b–61 Speech of Aeneas
362–64 Dido's emotions
365–87 Dido's speech
388–415 Narrative. Preparations for departure

The famous defense of Aeneas in 333b–61 is the focal point about which the corresponding passages revolve. However much modern readers may sympathize with Dido, it is evident that Vergil looked upon the queen as a danger to be resisted; Aeneas' duty lay elsewhere, and the poet has emphasized the rightness of Aeneas' decision by placing his speech in the very center of the book.

The third structural device is the use of tripartite divisions everywhere. Sinon in II makes three speeches (77–104, 108–44, 154–94), each with its appropriate effect on his listeners. The activity of Allecto in VII is threefold: she incites Amata (341–405),

[10] The numerical symmetry in VI.56–123 (21, 6, 15, 5, 21) resembles that which appears in several of the *Eclogues*, e.g. in I:

5 1–5 Introduction
21 6–26 Good fortune of Tityrus
13 27–39 Tityrus in Rome
6 40–45 Benefits from the youth (focal point)
13 46–58 Tityrus at home
20 59–78 Plight of Meliboeus
5 79–83 Conclusion

Cf. Richardson (above, note 9) 122 f.

[11] See Pease (above, note 4) 30.

Turnus (406–74), and the hounds of Ascanius (475–510), and the results of her actions are described in reverse order in 572–82 (shepherds, Turnus, Amata). In IX the tragic story of Nisus and Euryalus is a miniature drama in three parts: the scene in the Trojan camp (176–313), the slaughter in the camp of the enemy (314–66), their departure and death (367–449).

An examination of the twelve books of the *Aeneid* reveals that every book falls naturally into three main divisions,[12] and each main section is usually subdivided into three parts. Book II illustrates well Vergil's interest in tripartite structure:

A. 1–249. Sinon (57–194), Laocoon (40–56, 195–227), and the wooden horse (on shore, 1–39; in city, 228–49).[13]
B. 250–558. Return of Greeks (250–369), capture of Troy (370–505), death of Priam (506–58).
C. 559–804. Aeneas-Venus episode (559–633), Aeneas at home (634–729), departure and loss of Creusa (730–804).

In III the threefold division and the tripartite subdivisions are indicated even more clearly: we have nine episodes (or stops) on the journey from Troy to Sicily, divided into three main groups: Aegean area (1–191), western Greece (192–505) and Magna Graecia (506–718); the third episode in each group (Crete, 121–91; Helenus and Andromache, 294–505; the rescue of Achaemenides, 588–691) is more important and narrated at greater length.[14]

I add in outline form the tripartite divisions and subdivisions of Books IV (Tragedy of Love) and X (Tragedy of War):

[12] *Aeneid* V is often divided into two main sections: 1–603 (arrival and contests) and 604–871 (burning of the ships, departure, and death of Palinurus). But the *ludus Troiae* (545–603) was a spectacle, not a contest, and it was not included among the games announced by Aeneas in 64–70; also, it is the most Roman element in the book which begins the Roman and Augustan portion of the poem (V–VIII). If we look upon the *ludus Troiae* as a main division, V has the normal tripartite structure of the other eleven books: 1–544, arrival and contests; 545–603, *ludus Troiae*; 604–871, burning of ships and departure. The fact that the Trojan spectacle is short (59 verses) is not a valid reason against accepting it as a main division; the second section of VIII (370–453, the night scene of Venus and Vulcan and the making of the armor) is almost as short (84 verses).

[13] The three parts of the first main division are here arranged in a recessed panel pattern (a b c b a), with the story of Sinon as the focal point. For the division after 227, see A. Cartault, *L'Art de Virgile dans l'Énéide* (Paris 1926) 181, 184.

[14] See R. B. Lloyd, "*Aeneid* III: A New Approach," *AJP* 78 (1957) 136 ff. K. Büchner, *P. Vergilius Maro, der Dichter der Römer* (Stuttgart 1956) 336, with less plausibility views the journey to Italy (506–47) and Scylla and Charybdis (548–69) as one episode and lists Drepanum (707–15) as the ninth and final episode.

IV.	A.	1–172	Dido's love and its consummation
		1. 1–89	Growth of Dido's love
		2. 90–128	Juno-Venus scene
		3. 129–172	Hunting scene and "coniugium"
	B.	173–449	Aeneas' determination to leave
		1. 173–278	Fama—Iarbas—Jupiter—Mercury
		2. 279–415	Narrative; speeches of Dido, Aeneas, Dido; narrative
		3. 416–449	Attempted reconciliation fails
	C.	450–705	Aeneas' departure and Dido's suicide
		1. 450–552	Magic rites and Dido's lament
		2. 553–583	Aeneas' departure
		3. 584–705	Dido's curses and suicide
X.	A.	1–361	Return of Aeneas
		1. 1–117	Council of the gods
		2. 118–255	Return of Aeneas; catalogue of ships
		3. 256–361	Landing and battle
	B.	362–688	Death of Pallas
		1. 362–478	Aristeia of Pallas
		2. 479–605	Death of Pallas and effect on Aeneas
		3. 606–688	Removal of Turnus from battle
	C.	689–908	Deaths of Lausus and Mezentius
		1. 689–746	Aristeia of Mezentius
		2. 747–832	Death of Lausus
		3. 833–908	Death of Mezentius

It is important to note how frequently Vergil uses the central position in each book—the second subdivision of the second main division—for material of special significance. In II it is the capture of Troy (370–505), in III the stop at Actium (274–93)—an important Roman theme. We have seen above that the speeches of Aeneas and Dido are at the very center of IV and Aeneas' decision leads to Dido's suicide at the end of the book[15]; in like manner the death of Pallas in X.479–605 is the focal point of the entire book and is decisive for the death of Turnus at the conclusion of the poem; cf. XII.940 ff. These passages, so important for the action of the *Aeneid*, receive added emphasis from their central position in each book, a position resulting from Vergil's arrangement of his material into tripartite divisions and subdivisions.

[15] The meeting of Aeneas and Dido in the underworld (VI.450–76) is, appropriately, the second of the three encounters (Palinurus, Dido, Deiphobus) in the second main division of VI (236–547).

II

I have given above a brief analysis of the *Aeneid* as a whole and have discussed the structure of individual books for two reasons: (1) the mathematical proportions in the poem are achieved by the use of alternating, tripartite, and framework patterns very similar to and often identical with those already described, and (2) it was only when I had determined the tripartite divisions of the books that the mathematical ratios appeared. The main divisions were of very uneven length, and it occurred to me that these inequalities might be intentional. Then I made the amazing discovery (amazing to me, at any rate, as I had no idea what would develop) that the line totals of at least two of the three main divisions in each book (and often of all three) were always in the same approximate proportion; also, the subdivisions not only contained the same ratios but in most instances combined to produce a similar ratio in each main division. Additional investigation revealed that within the divisions and subdivisions were smaller narrative units composed of others still smaller (episodes and speeches), all in the same proportion, the exact or approximate Golden Section, or Golden Mean ratio; in other words, each book of the *Aeneid* is constructed on the basis of mathematical symmetry, with the proportions in the short passages combining into larger units until we reach the ratios in the subdivisions and main divisions of each book.

The Golden Section, famous in mathematics, in art and architecture, in aesthetic theory, is that ratio according to which the greater part is to the lesser as the sum of the two is to the greater. This is the extreme and mean ratio of Euclid 2.11 and 6.30. The two parts, greater and lesser, are usually called major and minor. If we denote the major by M and the minor by m, $M/m = (M+m)/M = 1.618$, and $m/M = M/(M+m) = .618$. These total $2.236 = \sqrt{5}$; $1.618 = 1/2(\sqrt{5}+1)$ and. $618 = 1/2(\sqrt{5}-1)$. In my calculations I have regularly divided the smaller number by the larger, as the mathematics is somewhat simpler, and the ratios which I give below result from the division of the major by the sum of major and minor, i.e. $M/(M+m)$.[16]

[16] I omit the ratio of m/M, as this can always be deduced from the slightly more accurate $M/(M+m)$; e.g. if the major is 75 lines and the minor is 46 lines, $M/(M+m) = 75/121 = .620$, a variation of .002 from the perfect .618; the variation in the ratio

We can reach the Golden Section most quickly by a mathematical series named after the thirteenth-century Italian mathematician Fibonacci, where each number is added to the second to produce the third, beginning with 1 and 1; i.e. 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, . . .; 21/34 or 34/55 or 55/89 = .618. But we can take any two numbers, 1 and 3, or 1 and 4, or 1 and 5, or 2 and 5, and these also produce series which lead eventually to the Golden Section, e.g. 1, 5, 6, 11, 17, 28, 45, 73, 118, 191, . . .; 118/191 = .618; or 2, 5, 7, 12, 19, 31, 50, 81, 131, . . .; 81/131 = .618. Here we necessarily go to much higher numbers than in the case of the Fibonacci series.[17]

If Vergil writes an episode with two passages of unequal length, e.g. of 25 and 75 lines, M/(M + m) = .750; if the two passages are 45 and 55 lines, M/(M + m) = .550; these are far from the Golden Section. But in the *Aeneid* the ratio in the short episodes and speeches, in longer narrative units, and in the subdivisions and main divisions of each book is almost always in the approximate area of .618. This cannot be chance. In a passage of 100 lines, even if we eliminate the extremes of 1–19 and 99–81, we have a possible range for M/(M + m) from .800 (80/100) to .500 (50/100); 61/100 produces a ratio of .610, 62/100 that of .620, and 63/100 that of .630. In a series from 20–80 to 80–20 these three ratios

m/M is about two and one-half times as great in the opposite direction, i.e. 46/75 = .613. It is therefore unnecessary to list both ratios. On this and other matters connected with the Golden Section, I am deeply indebted for advice and assistance to Professor William Feller of the Department of Mathematics, Princeton University. Also, although I frequently refer to the perfect .618, there is of course no such thing; $\frac{1}{2}(\sqrt{5}-1)$ is an irrational number which approaches .618034. Since I carry the ratios only to the third decimal, I use the terms "exact .618" and "perfect .618" to distinguish the more exact ratio from approximate ratios such as .610 or .625.

[17] On the Golden Mean ratio and the Fibonacci series, see R. C. Archibald, "Golden Section" and "A Fibonacci Series," *Amer. Math. Monthly* 25 (1918) 232–38, reprinted with corrections and additions in J. Hambidge, *Dynamic Symmetry. The Greek Vase* (New Haven 1920) 152–57; D. W. Thompson, "Excess and Defect: or the Little More and the Little Less," *Mind* 38 (1929) 43–55, reprinted in *Science and the Classics* (Oxford 1940) 188–213; cf. also V. Capparelli, "Ludus Pythagoricus e divina proporzione," *Sophia* 26 (1958) 197–210. On the Fibonacci series in nature, e.g. in the spirals of fir-cones and flowers (phyllotaxis), cf. D. W. Thompson, *On Growth and Form*[2] (Cambridge 1942) 2.921–33. For the use of the Golden Section in art and architecture, see M. C. Ghyka, "The Pythagorean and Platonic Scientific Criterion of the Beautiful in Classical Western Art," in F. S. C. Northrup (ed.), *Ideological Differences and World Order* (New Haven 1949) 90–116; M. Borissavlievitch, *The Golden Number and the Scientific Aesthetics of Architecture* (New York 1958). Earlier items are listed by G. Le Grelle, S.J., "Le premier livre des *Géorgiques*, poème pythagoricien," *Les études class.* 17 (1949) 145, note 4.

each occur twice; the probability of their appearance is six out of sixty times or once in ten times. A range from .610 to .626, a variation of .008 from the perfect .618, would thus be extremely difficult to achieve accidentally, and a perfect .618 is impossible. And yet Vergil has many perfect ratios of .618, most of them in short passages of from 30 to 70 lines, and in most instances he reaches this perfect ratio by the numbers 13 and 21, or 21 and 34; i.e. he is using the numbers of the Fibonacci series. This series and its multiples, e.g. 26, 42, 68 ($=2 \times 13, 21, 34$) or 40, 65, 105 ($=5 \times 8, 13, 21$), appear in the *Aeneid* more than 300 times, and the next simplest Golden Mean series (1, 3, 4, 7, 11, 18, 29, 47, 76, . . .; $47/76 = .618$) occurs about 90 times. The other series, such as 1, 4, 5, 9, 14, 23, . . . or 1, 5, 6, 11, 17, 28, . . . likewise appear, but only 15 to 30 times each; in other words, the Fibonacci series occurs about three and one-half times as often as 1, 3, 4, 7, 11, 18, . . . and this in turn is three times as frequent as any of the others.

This would seem to rule out not only chance but also intuition or poetic instinct. Friends with whom I have discussed Vergil's use of the Golden Section suggest that he had a feeling for this particular ratio but that he did not deliberately count lines. But when these various mathematical series appear again and again, with the simplest and quickest series used so much more frequently than the others, I cannot believe that we have here merely a subconscious desire for proportion and symmetry. It seems far safer to assume that Vergil purposely introduced Golden Mean ratios into all parts of the *Aeneid* by means of these mathematical series. We must keep in mind Suetonius' statement in his *Life* of Vergil (15) that the poet as a student gave special attention to the study of mathematics.

III

Before I present typical examples of the proportions in the *Aeneid*, I wish to point out that the same Golden Mean ratio is found in both the *Eclogues* and the *Georgics*. The first to reveal Vergil's knowledge and use of the Golden Section was Father Le Grelle who in 1949 explained the troublesome structure of Book I of the *Georgics* by means of the numerous ratios which he found therein.[18] Le Grelle accepts the division of the book into

[18] Le Grelle (above, note 17) 139–235.

"Works" (43–203 = 161 lines) and "Days" (204–463a = 259.5 lines); these parts as minor and major produce the Golden Section; M/(M + m) = 259.5/420.5 = .617.[19] The prologue (5b–42) and the epilogue (463b–514) combine to form a major of 89 lines, which, with the central astronomical passage ("foyer astronomique," 204–58) of 55 lines as minor, reveals the perfect Golden Section: 55/89 = 89/144 = .618. We have here a tripartite framework design with *a* and *c* enclosing *b*, in the pattern b/(a + c), and the totals are higher numbers in the Fibonacci series, . . . 13, 21, 34, 55, 89, 144, . . . In each of numerous smaller units, which he terms "chrysodes," he likewise finds a major and a minor part which are in proportion, e.g. 100–117 (18 = m) and 118–46 (29 = M) produce the ratio .617 (29/47), and 100–146 (47) is the major of a larger unit, with 147–75 (29) the minor; the ratio here is the exact .618 (47/76) and the numbers are those in the series 1, 3, 4, 7, 11, 18, 29, 47, 76, . . .

Le Grelle's discovery of the Golden Mean ratios in *Georgics* I is confirmed by the many similar examples which I have detected elsewhere in the *Georgics* and also in the *Eclogues*. These were brought to my attention purely by chance. While working on the *Georgics* for my original book, I happened to read the statement of Wili that less than half of the *Georgics* is composed of didactic material, the greater part consisting of "Bild und Reflexion," the descriptive passages praising Italy, country life, Octavian, etc.[20] Curiosity led me to check the statement. Wili is quite wrong, for the technical sections on farming have much the greater extent, the ratio being 1352 verses to 835.[21] It was amazing to find here the exact Golden Mean ratio; M/(M + m) = 1352/2187

[19] Le Grelle regularly divides the total of major and minor by 1.618 to derive the major part, i.e., 420.5/1.618 = 259.88, a variation of a fraction of a line from the actual major (204–463a = 259.5). Since my procedure throughout is to divide the major by the total in order to derive the Golden Mean ratio (in this instance .617, a variation of .001 from the perfect proportion), the range of the proportions in the area of .618 is more easily discerned.

[20] W. Wili, *Vergil* (München n.d.) 54.

[21] The descriptive passages are the following: I.5b–42, 125–46, 231–58, 463b–514; II.1–8, 136–76, 319–45, 380–96, 458–542; III.1–48, 242–83, 284–94, 339–83, 478–566; IV.1–7, 116–48, 315–558. The section about the rebirth of bees (IV.281–314), although actually the beginning of the Aristaeus story, is itself of a technical nature and is not included among the descriptive passages. Also, the introductory statement of the plan of the poem (I.1–5a) and the final recapitulation (IV.559–66) are properly classed with the didactic portions.

= .618.[22] The possibility of this being accidental (in a poem of 2187 verses) would be at least one chance in 500.

I next turned to many short passages in *Georgics* II, III, and IV, especially those of a non-technical nature, and again the Golden Section appeared; e.g. (1) the praise of country life in II.458–540 (four parts in a chiastic arrangement): country life (459–74) and the poet's ambition (475–89) as minor (32 lines), the *felix-fortunatus* passage (490–94) and the advantages of country life with the new Golden Age in Italy (495–540) as major (51 lines); 51/83 = .614; (2) the plague in III.478–566, which divides into three stages: 478–502, 503–36, and 537–66 (the third stage beginning with the reversal of nature—almost a Golden Age in reverse); the ratio is tripartite in a framework pattern, with the first and third stages, 478–502 and 537–66 (a total of 55 lines = M) enclosing the second stage, 503–36 (34 = m); 34/55 = 55/89 = .618, again the perfect Golden Section in the Fibonacci series 34, 55, 89; (3) the Aristaeus and Orpheus stories in IV.281–558: (a) Aristaeus, 281–452 (171) as M; (b) Orpheus and Eurydice, 453–529, and (c) the restoration of the bees, 530–58, as m (77 + 29 = 106); 171/277 = .617.[23]

The Golden Mean ratio appears likewise in the *Eclogues*, e.g. in X: 1–30 (introduction, nymphs, shepherds, Apollo, Pan) provide the minor, the major consisting of 31–77 (song of Gallus, conclusion); 47/77 = .610. The recessed panel construction of *Eclogue* I was analyzed above[24]; the focal point (40–45) and the two Tityrus passages which frame it (27–39, 46–58) provide the minor of 32 verses; the balancing passages at the beginning and the end of the poem (1–26 and 59–83) are the major of 51 verses; 51/83 = .614. Similarly, in II, the focal point of 28 lines (28–55, invitations to Alexis and gifts offered) is the minor and is framed by 1–27 (introduction; Corydon reproaches Alexis) and 56–73 (Corydon reproaches himself; conclusion) which provide the major of 45 lines; 45/73 = .616. In both I and II the patterns of

[22] These totals do not include IV.338, bracketed as an interpolation by Hirtzel and other editors. If we retain the verse, the ratio is not affected: 1352/2188 = .618.

[23] Each of these parts likewise contains a ratio: (a) 281–386 (105) as M, and 387–452 (66) as m; 105/171 = .614; (b) 453–98 and 528–29 (48) as M, and 499–527 (29) as m; 48/77 = .623; (c) 530–47 (18) as M, and 548–58 (11) as m; 18/29 = .621. In the Aristaeus story I omit verse 338; cf. above, note 22. The ratios listed here are composed of smaller units also in proportion; I have detected 22 ratios in IV.281–558.

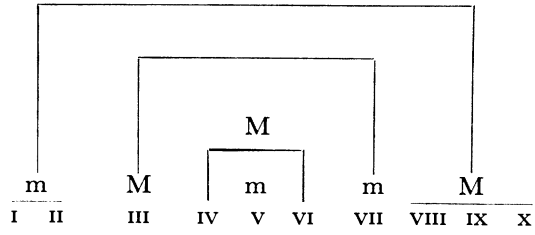
[24] See above, note 10.

thought, the recessed panel or framework construction, and the Golden Mean ratios are subtly interwoven.

Much has been written about the structure of the *Eclogues* as a group; especially significant are the reverse parallelism of I–IV and VI–IX about V as the central poem (with X, the poet as a shepherd, added to balance V, the shepherd-poet as a god), and the arrangement of the poems in triads with the central triad (IV–VI) on more cosmic themes and X combining the themes of all three triads.[25] Golden Mean ratios appear in the collection as a whole, and the corresponding poems may be grouped into major and minor units, as follows:

I+II=m (156); VIII+IX+X=M (253);	253/409=.619;
III=M (111); VII=m (70);	111/181=.613;
IV+VI=M (149); V=m (90);	149/239=.624.[26]

The existence of these proportions in the *Eclogues*, especially in the two corresponding responsive songs III and VII, and in IV–VI, where IV ("the world to come") and VI ("the world that was") enclose V (the death and deification of Daphnis) is most impressive. The relation between the major and minor poems may be diagrammed as follows:



If we view the *Eclogues* as composed of three triads and a final poem which blends the shepherds and the realism of Triads One and Three with the gods and the fantasy of Triad Two, the Golden Section is likewise present; the first two Triads (I–III, IV–VI) form a major of 506 lines, and the final Triad plus X a minor of 323 lines; 506/829=.610. The proportions thus support both the concentric arrangement of the *Eclogues* about V (Daphnis) and the triadic

[25] For discussion and bibliography, see Duckworth (above, note 3) 3 f.; cf. also E. A. Hahn, "The Characters in the *Eclogues*," *TAPA* 75 (1944) 239–41.

[26] For these three ratios I am indebted to Edwin Brown, now completing a Princeton doctoral dissertation, "Studies in the *Eclogues* and *Georgics* of Vergil."

structure, and they illustrate the complexity of Vergil's compositional patterns.

IV

Although Vergil's use of mathematical proportions in the *Eclogues* and the *Georgics* was now clearly evident, I still had no idea that the same exact or approximate Golden Mean ratio would appear everywhere in the *Aeneid* and, strange to say, it did not occur to me to look for it. When, however, I had completed my structural analyses of the *Aeneid*, with each book divided into three main divisions, and with these in most instances falling into three clearly-marked subdivisions, the unequal length of the main divisions aroused my interest; then I discovered that in every book at least two and often all three main divisions were in proportion; e.g.:

A. Two divisions in proportion:

III.1–191 (m = 191), 192–505 (M = 310.8);
310.8/501.8 = .619;[27]

IV.1–172 (m = 171.4), 173–449 (M = 275.2);
275.2/446.6 = .616;

VII.1–285 (M = 284.2), 641–817 (m = 175.8);
284.2/460 = .618;

IX.1–175 (m = 171.4), 176–449 (M = 273.4);
273.4/444.8 = .615;

X.1–361 (M = 358.8), 689–908 (m = 217.8);
358.8/576.6 = .622;

XII.289–696 (M = 405.2), 697–952 (m = 256);
405.2/661.2 = .613.[28]

B. All three divisions in proportion:

II.1–249 and 559–804 (M = 488.8), 250–558 (m = 307.9);
488.8/796.7 = .614;

IV.1–172 and 450–705 (M = 425.4), 173–449 (m = 275.2);
425.4/700.6 = .607;

[27] In establishing my line totals I have omitted bracketed lines which do not appear in the best MSS., and I have counted the half-lines as fractions to the nearest decimal, e.g. 5/12 as .4, 7/12 as .6; see below, 213. Partial lines are treated likewise when speeches or episodes end within a verse.

[28] In general I have followed the paragraphing in Hirtzel's Oxford text. Mackail (edition of 1930) seems correct, however, in indicating a paragraph after 288 rather than after 286 (Hirtzel) or 282 (Sabbadini,[2] 1937), and he is supported by several shorter proportions which end with 288 or begin at 289.

VI.1–235 and 236–547 (M = 545.6), 548–901 (m = 353.7);
545.6/899.3 = .607;

VIII.1–369 and 370–453 (M = 451.4), 454–731 (m = 276.6);
451.4/728 = .620;

X.1–361 (m = 358.8), 362–688 and 689–908 (M = 543.6);
543.6/902.4 = .602.

It is interesting to note that in five of the six even-numbered books (those of greatest significance), all three main divisions combine to produce the approximate Golden Section. In Books II and IV the second division is enclosed by the first and third in a framework pattern.

I next examined the main divisions (36 in the 12 books) and found that each contained a Golden Mean ratio and that this was usually produced by the already-determined subdivisions (most often tripartite) of each main section, e.g.:

I.223–296 (m = 74), 297–417 (M = 121); 121.0/195.0 = .621;

II.250–369 (m = 119.2), 370–505 and 506–558 (M = 188.7); 188.7/307.9 = .613;
559–633 and 730–804 framing 634–729 (M = 147.7), (m = 94.6); 147.7/242.3 = .610;

III.506–547 and 548–587 588–718 (m = 81.4), (M = 129.6); 129.6/211.0 = .614;

IV.173–278 (m = 105), 279–415 and 416–449 (M = 170.2); 170.2/275.2 = .618;

VII.286–322 and 540–640 framing 323–539 (m = 138), (M = 216); 216.0/354.0 = .610;[29]

VIII.454–596 and 597–625 626–731 (M = 170.6), (m = 106); 170.6/276.6 = .617;[30]

IX.450–589 (m = 137.8), 590–671 and 672–818 (M = 228.1) 228.1/365.9 = .623;

[29] See below, 209, for a detailed analysis of this division.

[30] It is strange that most editors fail to indicate a paragraph after 596, which ends the first subdivision of the third and final portion of Book VIII. Fairclough in his Loeb edition (1916) begins a paragraph with 597 in both text and translation, as do Humphries (1951) and Guinagh (1953) in their translations. The departure from Pallanteum ends with 596; 597 ff. describe the arrival near Caere, many miles and certainly several hours on horseback from the site of Rome. The division after 596 is supported by the ratio in the third main section of the book and also by several shorter proportions.

x.1–117 and 256–361 framing 118–255	(M = 220.8), (m = 138);	220.8/358.8 = .615;[31]
362–478 and 606–688 enclosing 479–605	(M = 200), (m = 125.8);	200.0/325.8 = .614;
xi.1–99 and 100–138 139–224	(M = 138) (m = 86);	138.0/224.0 = .616.

As I worked from the main divisions and subdivisions to shorter narrative units, more and more Golden Mean proportions appeared. As I shall show below, my line-divisions are not arbitrary but are based upon the natural units of speech and action. I have made no attempt to discover every single ratio in the *Aeneid*, but my completed tables, arranged by patterns, contain 1044 proportions, from 75 to 100 a book, ranging from .60 to .636, and of these 622, about 60 per cent, are in the area from .610 to .626, i.e. within .008 of the exact .618. Every verse of the *Aeneid* (except the interpolations) appears in at least one, and usually in three, four, or more proportions. Moreover, 300 ratios are in what I consider an almost perfect area, from .615 to .621, and 45 are exactly .618.

Suetonius (*Vita* 23) states that Vergil made a prose outline of the *Aeneid* arranged in twelve books and then wrote *particulatim*—in small sections—as he pleased, taking nothing in order. These small sections are his units of composition, and they all contain exact or approximate Golden Mean ratios. As in the case of the main divisions and subdivisions listed above, Vergil achieves the Golden Section by a variety of patterns. I shall present sample ratios of each type and show how the units of speech and narrative combine in the proportions:

1. Two passages, *a* and *b*, with *a* either major or minor:

i.	8–33	(a) Juno's hostility	(m = 26),
	34–75	(b) Juno's lament and speech to Aeolus	(M = 42); 42/68 = .618;
	34–75	(a) Juno's lament and speech to Aeolus	(M = 42),
	76–101	(b) Aeolus' reply, the storm and Aeneas' words	(m = 26); 42/68 = .618;

[31] Mackail seems correct in beginning a new paragraph with 256 rather than 260 (as do Hirtzel and Sabbadini); see Cartault (above, note 13) 725, 727.

227–253	(a) Venus' complaint to Jupiter	(m = 27),
254–296	(b) Jupiter's prophecy	(M = 43); 43/70 = .614;
586–595a	(a) Aeneas appears before Dido	(m = 9.4),
595b–610a	(b) Aeneas' speech	(M = 15.2); 15.2/24.6 = .618;
II.506–525	(a) Priam and Hecuba at the altar	(m = 20),
526–558	(b) Priam slain by Pyrrhus	(M = 33); 33/53 = .623;
567–574	(a) Aeneas sees Helen	(m = 8),
575–587	(b) Aeneas thinks of killing Helen	(M = 13); 13/21 = .619;[32]
679–686	(a) the first omen	(M = 8),
687–691	(b) Anchises' words	(m = 5); 8/13 = .615;
IV.160–164	(a) the storm	(m = 5),
165–172	(b) the cave and the "coniugium"	(M = 8); 8/13 = .615;
522–533	(a) Dido's last night	(m = 11),
534–552	(b) Dido's lament	(M = 19); 19/30 = .633;
672–692	(a) Anna's lament	(M = 21),
693–705	(b) Iris releases Dido's spirit	(m = 13); 21/34 = .618;[33]
VI.789b–800	(a) Augustus and the Golden age	(M = 11.6),
801–807	(b) Augustus compared to Hercules and Bacchus	(m = 7); 11.6/18.6 = .624;
VII.323–329	(a) description of Allecto	(m = 7),
330–340	(b) Juno's instructions	(M = 11); 11/18 = .611;
670–677	(a) Catillus and Coras from Tibur	(m = 8),
678–690	(b) Caeculus from Praeneste	(M = 13); 13/21 = .619;

[32] This passage concerning Helen (II.567–87) is the minor (21 lines) of a larger proportion, the major consisting of 588–623, the Venus-Aeneas scene (34.8 lines); 34.8/55.8 = .624. The existence of these two ratios is an indication of the authenticity of the much discussed Helen passage; see below, 215, and note 57.

[33] Sabbadini begins a new paragraph with 672.

IX.590–620	(a) the taunts of Numanus	(m = 31),
621–671	(b) Numanus slain by Ascanius	(M = 51); 51/82 = .622;
X.791–816	(a) conflict between Aeneas and Lausus	(M = 26),
817–832	(b) Lausus' death and its effect on Aeneas	(m = 16); 26/42 = .619;
888–895	(a) Mezentius' horse slain	(m = 8),
896–908	(b) death of Mezentius	(M = 13); 13/21 = .619;
XII.919–939	(a) the wounded Turnus appeals for mercy	(M = 21),
940–952	(b) Aeneas sees the sword-belt of Pallas and slays Turnus	(m = 13); 21/34 = .618.[34]

Vergil's use of the Fibonacci series, . . . 5, 8, 13, 21, 34, . . . (or multiples thereof) is seen in many of the ratios listed above. It is worth noting that the most perfect ratios appear usually at the beginning and the end of the books; this suggests that Vergil wishes the attentive listener to be conscious of the mathematical symmetry of the poem and perhaps to detect more readily the approximate ratios elsewhere in each book. Or the hearer may be expected to derive a subconscious pleasure from the harmony of the proportions.

[34] Modern editors punctuate with a semicolon at the end of 939, but a full stop seems advisable as it places more stress on the *clementia* of Aeneas in 940, before he sees the swordbelt. (See above, 186 f., on the close relationship between the conclusion of XII and that of VI, where Anchises in 853 recommends both *clementia* and *iustitia*.) The beginning of 940: *et iam iamque* as a new sentence is recommended also by the fact that editors so frequently start a new sentence (and often a new paragraph) with *et iam* and *iamque*; for *et iam* (Hirtzel's text), cf. Aen. I.223, 302; II.254; IV.584, etc.; for *iamque*, cf. I.419, 695; II.132, 567, 730, 789, 801; III.135, 356, 521. etc. Finally, for three hundred years prior to 1775, almost every Vergil edition (including the Aldine, Juntine, Elzevir, and Ruaeus texts) had a full stop at the end of 939, and in several (e.g. the editions of 1533 and 1540 by R. Stephanus and the Venice editions of 1539, 1555, 1566, and 1582) line 940 began not only a new sentence but also a new paragraph. The punctuation with a full stop after 939 is found as late as the mid-nineteenth century, cf. e.g. F. Bowen (Boston 1842), J. G. Cooper[12] (New York 1863) E. Benoist (Paris 1872). The semicolon first appears regularly in Heyne's editions (1775, 1789, 1793), is adopted by Wagner, Forbiger, Conington, and Ribbeck in the nineteenth century, and accepted by present-day editors. There seems no reason to follow Heyne here; both structure and sense favor the punctuation of the earlier editors. Perhaps in the future it might even be advisable to begin a new paragraph with 940; this would give proper emphasis to Aeneas' thought of *clementia*.

The use of two passages to produce Golden Mean ratios occurs less frequently than that of three or more.

2. Tripartite (non-framework): three passages, *a*, *b*, and *c*, with a+b or b+c either major or minor. Actually, this is a variation of the first, or bipartite, pattern, with *a* or *b* subdivided into two parts, episodes or speeches. E.g.:

I. 34–39	(a) Juno's lament	(m = 16),
50–64	(b) description of Aeolus	
65–75	(c) Juno's words to Aeolus	(M = 26); 26/42 = .619;
76–80	(a) Aeolus' speech	
81–91	(b) the winds released	(M = 16),
92–101	(c) Aeneas' lament	(m = 10); 16/26 = .615;[35]
586–612	(a) Aeneas appears before Dido and speaks	(m = 27),
613–630	(b) Dido's welcome	
631–656	(c) banquet preparations and gifts	(M = 43.6); 43.6/70.6 = .618;
723–735	(a) conclusion of banquet and Dido's words	(m = 13),
736–747	(b) song of Iopas	
748–756	(c) Dido urges Aeneas to speak	(M = 21); 21/34 = .618;
II. 730–744	(a) departure and loss of Creusa	
745–770	(b) search for Creusa	(M = 40.2),
771–795	(c) vision of Creusa	(m = 24.7); 40.2/64.9 = .619;[36]
III. 472–481	(a) farewell of Helenus	
482–492	(b) Andromache's farewell	(M = 21),
493–505	(c) Aeneas' parting words	(m = 13); 21/34 = .618;
IV. 90–104	(a) speech of Juno	(m = 15),
105–114a	(b) Venus' reply	
114b–128	(c) Juno's second speech	(M = 24); 24/39 = .615;
	Dido's speech to Anna in 416–436:	
416–423	(a) Aeneas trusts Anna	(m = 8),

[35] These two passages (I.34–75 and 76–101) are the major and minor respectively of the second proportion listed under the bipartite pattern.

[36] Both Sabbadini and Mackail begin a new paragraph with 730.

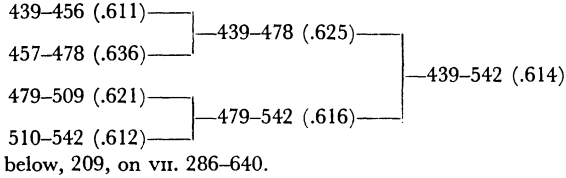
424–428	(b) Anna should entreat Aeneas	
429–436	(c) Dido wants Aeneas' departure postponed	(M = 13); 13/21 = .619;[37]
VI.756–807	(a) Alban kings, Romulus, Augustus	(m = 52),
808–853	(b) Roman kings and heroes	
854–892	(c) Marcellus passage	(M = 84.7); 84.7/136.7 = .620;
X.439–478	(a) Turnus against Pallas	(m = 40)
479–509	(b) death of Pallas	
510–542	(c) effect on Aeneas	(M = 63.6); 63.6/103.6 = .614.[38]

3. Tripartite framework: three passages, *a*, *b*, and *c*, with a + c as major or minor enclosing b, the focal point. The ratios in this category are twice as numerous as those in the second group. The focal point (b) may be a speech framed by two other speeches or by two episodes, or it may itself be narrative, but in most instances it is a passage of special significance; e.g.:

I.335–370a	(b) speech of Venus, enclosed by	(M = 35.4),
326–334	(a) speech of Aeneas	
372–385a	(c) speech of Aeneas	(m = 22.6); 35.4/58 = .610;[39]
III.320–343	(b) Andromache's speech, framed by	(m = 23.4),

[37] The ratio is the same if we view a + b as the major, but the shift in thought seems greater at line 424. Also, it should be noted that 424–28 is minor both to 416–23 and 429–36; these smaller units produce two additional ratios of .615 (8/13).

[38] As is regularly the case with larger units such as these, several smaller proportions combine to form the larger ratio. The smaller units in x.439–542 may be charted as follows (with the ratios in parentheses):



See below, 209, on VII. 286–640.

[39] Venus' speech in 335–70a itself subdivides into a framework pattern, with 343–64 (b), story of Dido, Sychaeus, and Pygmalion (M = 22), framed by 335–42 (a), introduction, and 365–70a (c), conclusion (m = 13.4); 22/35.4 = .621. The thought in 365 f. (*nunc ingentia cernes moenia surgentemque novae Karthaginis arcem*) takes us back to the first section (cf. 338: *Punica regna vides, Tyrios et Agenoris urbem*).

294–319	(a) Aeneas meets Andromache	
344–355	(c) Helenus welcomes Aeneas	(M = 37.6); 37.6/61 = .616;
IV.331–361	(b) Aeneas' defense, framed by	(m = 30.6),
303–330	(a) Dido's speech	
362–387	(c) Dido's speech	(M = 52); 52/82.6 = .630;
648–671	(b) final speech and suicide of Dido, enclosed by	(m = 24),
630–647	(a) Dido prepares for suicide	
672–692	(c) lament of Anna	(M = 39); 39/63 = .619;
VI. 77–103a	(b) description of Sibyl and prophecy, enclosed by	(m = 26.2),
56–76	(a) speech of Aeneas	
103b–123	(c) speech of Aeneas	(M = 41.4); 41.4/67.6 = .612;
868–886a	(b) Anchises' speech about Marcellus, framed by	(m = 18.2),
854–867	(a) Aeneas sees Marcellus	
886b–901	(c) instructions and departure	(M = 29.8); 29.8/48 = .621;
VIII.337–358	(b) Evander's description of early Rome, enclosed by	(m = 22),
313–336	(a) history of Latium	
359–369	(c) Aeneas at Evander's home	(M = 35); 35/57 = .614;
675–713	(b) battle of Actium, framed by	(m = 39),
626–674	(a) scenes of history of Rome; ocean	
714–728	(c) triumphs of Augustus	(M = 64); 64/103 = .621;

IX.324–356 (b) aristeia of Nisus and Euryalus, (M = 33),
enclosed by
314–323 (a) in the camp of the enemy; Nisus suggests slaughter
357–366 (c) taking of spoils (m = 20); 33/53 = .623;

XII. 18–45a (b) speech of Latinus, (M = 27.2),
framed by
10–17 (a) speech of Turnus
45b–53 (c) speech of Turnus (m = 16.8);
27.2/44 = .618.[40]

4. Four or five passages, usually interlocked, with b + d as major or minor alternating with a + c or a + c + e. Both major and minor passages are not infrequently bound together by similarity of theme or identity of speaker; e.g.:

IV.129–172, the hunting scene:
129–135 (a) preparation for hunt
136–150 (b) description of Dido and Aeneas
151–159 (c) the hunt
160–172 (d) the storm and the "coniugium"
The two Dido and Aeneas passages (b + d) provide the major (28), the scenes concerning the hunt the minor (16); 28/44 = .636;

279–415, the central core of Book IV, discussed above:
279–304 (a) preparation for departure
305–330 (b) Dido's speech
331–361 (c) Aeneas' defense
362–387 (d) Dido's response
388–415 (e) preparation for departure
The Aeneas passages (a + c + e) form the major (84.2), the Dido speeches (b + d) the minor (52); 84.2/136.2 = .618;[41]

[40] The totals in this ratio (16.8, 27.2, 44) are numbers in the Fibonacci series: 21, 34, and 55 × .8, hence the exact Golden Section.

[41] This passage is so symmetrically constructed that the approximate Golden Section also appears if we view the three speeches (b + c + d) as major (82.6) and the framing passages concerning Aeneas' departure (a + e) as minor (53.6); 82.6/136.2 = .606.

VII.406–474, the Allecto-Turnus scene:
406–434 (a) Allecto visits Turnus and speaks
435–444 (b) reply of Turnus
445–457 (c) action and words of Allecto
458–474 (d) effect on Turnus
The two passages of the major (a + c = 41.6) concern Allecto, those of the minor (b + d = 26.4) concern Turnus; 41.6/68 = .612;

IX.184–223, alternating speeches:
184–196 (a) speech of Nisus
197–206 (b) speech of Euryalus
207–218 (c) speech of Nisus
219-223 (d) speech of Euryalus
The two speeches of Nisus are the major (a + c = 25), those of Euryalus the minor (b + d = 15); 25/40 = .625.

The amazing feature about the ratios in this pattern is that the major (a + c, or b + d) is so often composed of speeches which are linked together by similarity of theme or identity of speaker, or by the second speech being an answer to or a development of the first; e.g.:

II.650–678 (four-part); M = 18:
657–670 (b) Aeneas' speech
675–678 (d) Creusa's words; 18/29 = .621;

III. 84–101 (four-part); M = 11:
84–89 (a) Aeneas' prayer to Apollo
94–98 (c) reply of oracle; 11/18 = .611;

472–492 (five-part); M = 13:
475–481 (b) Helenus' farewell
486–491 (d) Andromache's farewell; 13/21 = .619;

V.700–745 (five-part); M = 28:
708–718 (b) advice of Nautes
724–740 (d) words of Anchises; 28/46 = .609;

VIII.370–406 (five-part); M = 22.6:
374–386 (b) Venus' words to Vulcan
395–404a (d) Vulcan's words to Venus 22.6/37 = .611;

IX.224–280a (four-part); M = 35.4:
234–245 (b) Nisus to Trojans
257–280a (d) Ascanius to Nisus; 35.4/56.4 = .628;

x.846–871	(four-part); M = 16.2:	
846–856a	(a) Mezentius reproaches himself	
861–866	(c) Mezentius speaks to his horse;	16.2/26 = .623;
xii.134–160	(four-part); M = 17:	
142–153	(b) Juno to Juturna	
156–160	(d) Juno to Juturna;	17/27 = .630.

The relative frequency of the four patterns illustrated above is as follows: (1) bipartite, 394; (2) tripartite (non-framework), 138; (3) tripartite framework, 276; and (4) four or more parts, usually interlocking, 236—a total of 1044 ratios.[42] Also, in the *Aeneid* as a whole, i–iv and vii–x are the major (a + c = 6223.4) and v–vi and xi–xii the minor (b + d = 3628.8); 6223.4/9852.2 = .632.[43] In each half the first four books are the major, the last two the minor, and if we view the *Aeneid* as a trilogy, with i–iv and v–viii the major, another approximate Golden Mean ratio appears.

I said above that Vergil's use of the Fibonacci and the other mathematical series seems convincing evidence that the Golden Mean ratios throughout the *Aeneid* could not be the result of intuition or a subconscious desire for this particular proportion, but reveal a deliberate arrangement of the parts of each book. This conclusion is supported by the fact that in every book there is an amazing correlation between the narrative units, the short passages with ratios, and the component parts of the main divisions. As a typical example I give the second main division of Book vii with its tripartite subdivisions, and then present a chart of the passages containing proportions (with the ratios in parentheses).[44]

[42] Whereas I believe that Vergil deliberately introduced the Golden Section into all parts of the *Aeneid*, I do not maintain that he was conscious of the existence of all the 1044 ratios which I have detected. Many of these may have been, and undoubtedly were, the more or less accidental result of ratios in smaller or larger passages; e.g. if two passages appear side by side with *a* as minor and *b* as major in each, the combined passages may be in proportion in the familiar interlocked four-part pattern, with a + c as minor and b + d as major; conversely, if we have the framework pattern with b as minor enclosed by a + c, *a* and *b* or *b* and *c* are often in proportion.

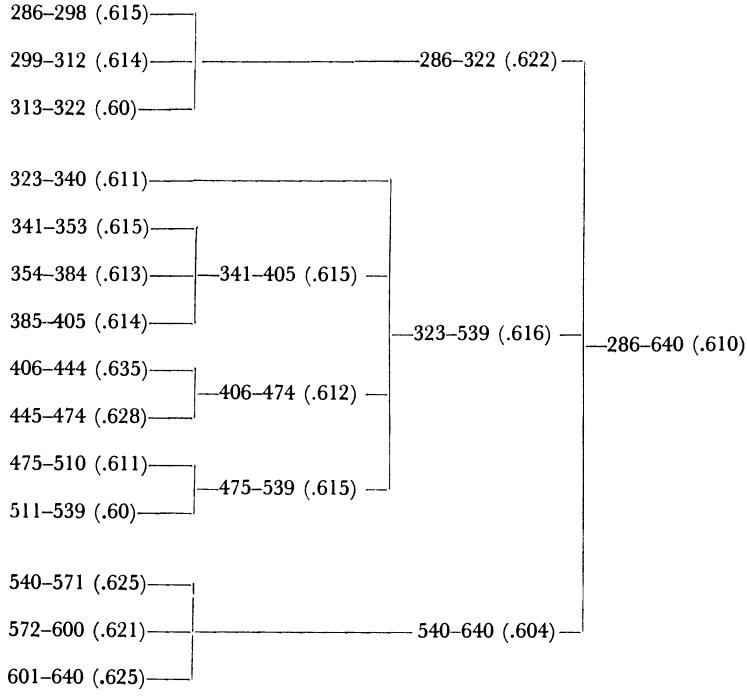
[43] The total lines of the *Aeneid* include 1.1a–1d; see below, 215.

[44] In the work of which this article is a summary I have prepared similar charts for each book of the *Aeneid*, with the passages in each chart keyed to the tables describing the various ratios.

VII.286–640:

(1) 286–322 Juno's lament
(2) 323–539 The Allecto episode:
  323–340 Juno summons Allecto
  341–539 Threefold activity of Allecto
    341–405 Maddens Amata
    406–474 Maddens Turnus
    475–539 Maddens hounds of Ascanius
(3) 540–640 Outbreak of the war

The ratios in 286–640 are as follows:



In this one main division I find 21 proportions, the smaller narrative units combining into larger passages until we reach the main division, 286–640. The ratio in this passage (.610) is derived from 323–539 as major, which is framed by 286–322 and 540–640 as minor, and these three passages are identical with the tripartite

subdivisions listed above. This section of *Aeneid* VII illustrates the manner in which the narrative units, small and large alike, coincide with the Golden Mean proportions. When such a striking correspondence appears in almost every division of every book throughout the *Aeneid*, we can only conclude that Vergil intentionally composed his narrative units in mathematical ratios and used them as building blocks to form the major or minor parts of the larger proportions.

V

We have seen *how* Vergil achieves the Golden Section by means of bipartite, framework, and alternating patterns and the extent to which he carefully combines structure and content in the proportions. To explain *why* Vergil composed his narrative by mathematical symmetry is more difficult—unless, with many Vergilian scholars, we believe that he was a Neopythagorean. Some, like Le Grelle and Maury, find number symbolism in both the *Georgics* and the *Eclogues*, especially in the use of the mystical numbers 333 and 666; Carcopino sees Pythagoreanism everywhere in the Fourth *Eclogue*; Boyancé and others believe that the *felix* passage in *Georg.* II.490 ff., refers not to Lucretius but to Pythagoras; still others find Pythagoreanism in *Aeneid* VI.[45]

If we could accept the view that Vergil was a Pythagorean, his interest in—almost obsession with—mathematical ratios could be explained more readily; but I do not believe that his use of the Golden Section proves his Pythagoreanism, for a very simple reason: *he was not the only Roman poet to compose by Golden Mean ratios.* Catullus 64, the wedding of Peleus and Thetis enclosing the story of Ariadne, contains 31 examples of the Golden Section.[46] The

[45] See Le Grelle (above, note 17), especially 152 ff., 159 ff., 222 ff.; P. Maury, "Le secret de Virgile et l'architecture des Bucoliques," *Lettres d'humanité* 3 (1944) 71–147, especially 111 ff.; J. Carcopino, *Virgile et le mystère de la IV^e Églogue*[2] (Paris 1943); P. Boyancé, "Sur quelques vers de Virgile (*Géorgiques*, II, v. 490–492)," *Rev. Arch.* 25 (1927) 361–79; P. Scazzoso, "Reflessi misterici nelle 'Georgiche' di Virgilio," *Paideia* 11 (1956) 5–28.

[46] For the three main divisions of the poem I follow C. Murley, "The Structure and Proportion of Catullus LXIV," *TAPA* 68 (1937) 308: 1–51, wedding of Peleus and Thetis; drapery; 52–250, story of Ariadne; 251–408, drapery; wedding of Peleus and Thetis. 1–51 and 52–250 provide the major; 250/408.=.613. The ratios in the main divisions are tripartite framework, as follows: (a) 12–30, love of Thetis and

books of Lucretius, in spite of the uncertainty of the text, are in proportion with I–IV the major (4675) and V–VI the minor (2747); 4675/7422 = .630; each book divides into four or five main divisions which produce ratios ranging from .608 to .620.[47] I have examined Book I in detail; it contains 62 ratios with a correlation between the proportions and the units of thought very similar to that appearing everywhere in the *Aeneid*. The structure of Horace's *Ars poetica* has been much debated, but there is general agreement that 1–294 concern poetry and 295–476 the poet and critic.[48] The major divided by the total, i.e., 294/476, equals .618, the perfect Golden Section, and the totals, 182, 294, 476, are multiples (by 14) of 13, 21, 34, the Fibonacci series which we find so often in the *Aeneid*.[49] Many Golden Mean ratios appear elsewhere in Horace's hexameter poems, e.g. *Sat.* I.10: 1–35, criticism of Lucilius (m = 35) and 36–92, Horace's own ideal (M = 57); 57/92 = .620. If we retain the disputed eight verses which most editors reject, the ratio changes to .570 (57/100); the presence of the Golden Section in the tenth satire thus strengthens the view that the initial eight verses are not by Horace. An interesting illustration of the four-part interlocked pattern occurs in *Sat.* II.3.82–295, where four vices are described: (a) avarice, 82–157, (b) ambition, 158–223, (c) self-indulgence including that of lovers (224–80), and (d) superstition (281–95). It is appropriate that avarice and self-indulgence, the two vices condemned

Peleus (m = 19), enclosed by 1–11 and 31–51 (M = 32); 32/51 = .627; (b) 116–237, lament and curse of Ariadne; earlier speech of Aegeus (M = 122), framed by 52–115 and 238–50 (m = 77); 122/199 = .613; (c) 323–83, song of the Parcae (m = 60), enclosed by 251–322 and 384–408 (M = 98); 98/158 = .620.

[47] E.g. Book I; I use Bailey's text of 1947, adding one line for each lacuna indicated. Leonard and Smith (1942) seem correct in viewing 921–50 as the introduction to the final division: 1–145 (a) and 635–920 (c) as minor (434), and 146–634 (b) and 921–1117 (d) as major (687) in a four-part interlocking pattern; 687/1121 = .613. In Book III (Bailey's divisions), 1–93 (a) and 94–416 (b) are the minor (417), with 417–829 (c) and 830–1094 (d) the major (677); 677/1094 = .619. Also in III the first three divisions are in proportion in a framework pattern, with 1–93 (a) and 417–829 (c) as major (505); 505/829 = .609; also 417–829 (c) is the major (412), with 830–1094 (d) the minor; 412/677 = .609.

[48] For discussion and earlier analyses, see W. Wili, *Horaz und die augusteische Kultur* (Basel 1948) 316 (and note 2), 325; G. Stégen, *Les Épîtres littéraires d'Horace* (Namur 1958) 8, 166 ff.

[49] This ratio in the *Ars poetica* was discovered also by K. Gantar, "De compositione Horatii 'Epistulae ad Pisones,'" *Živa Antika* 4 (1954) 277, but he does not mention the presence of the Fibonacci series.

by Horace so frequently elsewhere in his poetry, should form the major (a + c = 133); 133/214 = .621.

Mathematical symmetry based on the Golden Section thus seems a regular feature of the structure of Roman poetry in the first century B.C. In Book I of Lucretius I find an average of one ratio to every 18 verses, in Catullus 64, one to every 13.2 lines, but in Vergil's *Aeneid* one to every 9.4 lines. Vergil, with his interest in mathematics, has apparently carried the use of the Golden Section much farther than have the other Latin poets. But why?

The Golden Section has been viewed by many modern artists as one of the secrets of beauty and has played an important role in attempts to reduce beauty of proportion to a mathematical formula.[50] Almost a hundred years ago the German philosopher and psychologist, Gustav Theodor Fechner, conducted a most interesting experiment. By a kind of "pre-Gallup" poll he requested opinions, both likes and dislikes, from a large number of men and women about the innate beauty in a series of rectangles of different shapes. One rectangle had a far greater appeal than any other—absolutely no rejections and 35 per cent of the preferences—and this was the rectangle constructed on the Golden Section, with its sides 21 to 34, giving the exact ratio .618; the two rectangles closest to the "Golden Rectangle" each received over 19 per cent of the preferences.[51] Thus the "Golden Rectangle" and the two nearest to it in proportion received 74 per cent of the votes. This implies that the Golden Mean ratio has the most beauty on its own merits.

Did Vergil likewise believe that poetic passages or groups of passages bearing this same ratio, exact or approximate, had a mathematically formal beauty which could contribute to the perfection of his epic structure? I can think of no other explanation which will account for the presence of the Golden Section everywhere in the *Aeneid* and elsewhere in his poetry.

VI

Finally, are the proportions of significance for the text of the *Aeneid*? This is the most important question of all, and, in my

[50] Cf. G. Sarton, *A History of Science. Ancient Science through the Golden Age of Greece* (Cambridge [Mass.] 1952) 443; H. Weyl, *Symmetry* (Princeton 1952) 72; C. Ottaviano, "Nuove ricerche intorno all'essenza del bello," *Sophia* 22 (1954) 3–46.

[51] See G. T. Fechner, *Vorschule der Aesthetik*[3] (Leipzig 1925) 1.184–202.

opinion, the answer is a strong affirmative. The dozens and hundreds of Golden Mean ratios in the poem provide us with a control for a number of textual matters, and in this article I can enumerate these only briefly and in outline form; I shall give only a few of the many examples which illustrate each of the following categories.

1. Half-lines, such as we find in the *Aeneid*, appear in no other hexameter poetry, Greek or Roman. Some Vergilian scholars believe that these, or at least many of them, were deliberate, being rhetorically or emotionally effective[52]; but most think that Vergil in his final revision would have replaced them all by whole lines.[53] In determining the ratios, was I to count the hemistichs as fractions or as whole lines? Since the *Aeneid* was meant to be heard, the half-lines would strike the ear as fractions of a line, and I therefore counted them as such to the nearest decimal, .2, .3, .4, .6, or .7. Then I discovered that dozens of proportions were more accurate with the half-lines so treated and included several perfect .618 ratios. E.g. in v.315–26 (m = 318–22), with 322 (*tertius Euryalus*) as .4, the ratio is .614 (7/11.4) but with 322 as a whole line, we have an impossible .583 (7/12); similarly, in x.1–17 (M = 6–15, Jupiter's speech), with 17 (*pauca refert*) as .2, the ratio is .617 (10/16.2), but with 17 as a whole line it changes to .588 (10/17). VIII.469 (*rex prior haec* = .2) appears in three short proportions (454–80, 454–93, 469–519) with ratios .618, .617, .618 respectively; with 469 as a whole line, the ratios are less exact: .630, .625, .608.

On this mathematical basis about three-fourths of the half-lines may be viewed as intentional. The remaining hemistichs not only provide more exact ratios if we count them as whole lines but also, when they are so treated, the major and the minor in these passages become numbers in the Fibonacci and the other Golden Mean series which appear so often elsewhere in the proportions in the *Aeneid*. These half-lines would probably have been replaced by whole lines in the final revision; e.g.:

[52] Cf. e.g., J. Sparrow, *Half-lines and Repetitions in Virgil* (Oxford 1931) 23–46; W. W. Fowler, *Virgil's "Gathering of the Clans"*[2] (Oxford 1918) 93 f.

[53] See O. Walter, *Die Entstehung der Halbverse in der Aeneis* (Giessen 1933) 67; Büchner (above, note 14) 404.

Half-lines	*Value as fraction*	*Passage with ratio*	*Ratio as half-line*	*Ratio as whole line*	*Numbers in Golden Mean series*
II.640	.4	634–49	.610	.625	3, 5, 8 (x 2)
		624–49	.606	.615	5, 8, 13 (x 2)
		624–91	.623	.618	13, 21, 34 (x 2)
II.787	.7	771–804	.614	.618	13, 21, 34
III.640	.2	613–54	.631	.619	8, 13, 21 (x 2)
VIII.41	.4	36–65	.613	.621	11, 18, 29
		18–65	.612	.617	18, 29, 47
VIII.536	.4	520–40	.608	.619	8, 13, 21[54]

It thus seems possible to determine mathematically, to some degree at least, both the half-lines which were deliberate and those which would have been revised.

2. The ratios are helpful also for the problem of interpolations and so-called spurious passages. They support the rejection of individual lines which do not appear in the best MSS. and which are therefore bracketed as interpolations on MS. evidence[55]; they do *not* support the rejection by editors of lines or groups of lines on subjective grounds. E.g. Mackail speaks of the clumsy phrasing and the needlessness of IV.256–58; these lines occur in passages with the following ratios: .610, .625, .610, .60; with the three verses removed, the ratios become .544, .595, .659, .584. Ribbeck brackets VIII.42–49a, joining the remainder of 49 to the half-line in 41; with this passage omitted, five ratios change from .613, .612, .617, .618, .608 to .500, .550, .586, .591, .592. The removal of the many passages suggested by Ribbeck, Mackail, and other editors produces like results and in most instances any approximation to the Golden Section disappears.

[54] The ratios with the half-lines as fractions are the ones which I list in my tables. The other hemistichs which would probably have been changed to whole lines are II.767; III.661; IV.516; V.574, 653, 815; VI.835; X.284, 876; possibly II.623. Such a revision would have increased the number of perfect and almost perfect Golden Mean ratios.

[55] E.g. two ratios in which II.76 occurs move from .615 and .606 respectively to .583 and .593 if we retain the verse. The other lines to be rejected as interpolations are III.230; IV.273, 528; VI.242; VIII.46; IX.29, 121, 151, 529; X.278, 872; XII.612–13. I have omitted these verses in determining the ratios and have also ignored the second part of the line in V.595; XI.391, XII.218; these are more accurate when treated as hemistichs.

Certain passages are lacking in the MSS., but we have ancient testimony concerning their authenticity. Most editors reject I.1a–1d, the four verses preceding *arma virumque cano* which, according to Suetonius (*Vita* 42), were removed by Varius after Vergil's death; but Hirtzel seems correct in accepting them as genuine. I include these four lines in my proportions as their omission changes three ratios from .623, .631, .628 to .592, .663, .640.[56]

The famous Helen passage in II.567–88, omitted in the best MSS. but preserved by Servius, has been considered spurious by Heinze and others, but many scholars defend the passage as genuine.[57] This episode, itself a Fibonacci ratio of .619 (13/21) is also the minor of a proportion which of course vanishes if we reject the passage.[58] In II.559–663, the major (44.8) is 588–633, the minor (29) is 559–87; the ratio of .607 (44.8/73.8) becomes .848 (44.8/52.8) with the passage omitted. Every main division of every book in the *Aeneid* reveals the approximate Golden Section. But in the third main division of II (634–729, Aeneas at home, as minor, framed by 559–633, Helen and Venus episodes, and 730–804, departure from Troy), where we have a ratio of .610 (147.7/242.3), the removal of the Helen passage produces an impossible .571 (125.7/220.3). It seems hardly likely that 35 out of 36 main divisions would contain the Golden Section and that the one exception would appear in this much discussed passage. We have here a strong argument for retaining the Helen episode as authentic. It is possible that Vergil was dissatisfied with the passage and planned to revise it, but the substitute version would necessarily have been similar and of the same approximate length.

[56] The ratios do not support the retention of III.204a–204c, and VI.289a–289d, as Vergilian.

[57] Vergilian authorship is denied by R. Heinze, *Virgils epische Technik*[3] (Berlin 1915) 45 ff.; E. Norden, *P. Vergilius Maro. Aeneis, Buch VI*[3] (Leipzig 1926) 261 f.; H. Liebing, *Die Aeneasgestalt bei Vergil* (Kiel 1953) 44, 189 f. Those who defend the authenticity of the passage include H. R. Fairclough, "The Helen Episode in Vergil's *Aeneid* ii. 559–663," *CP* 1 (1906) 221–30; J. Gerloff, *Vindiciae Vergilianae. Quaestiones criticae de Aeneidis libri II 567–588* (Jena 1911); M. M. Crump, *The Growth of the Aeneid* (Oxford 1920) 44 ff.; F. W. Shipley, "The Virgilian Authorship of the Helen Episode, Aeneid II, 567–588," *TAPA* 56 (1925) 172–84; W. F. J. Knight, *Vergil's Troy* (Oxford 1932) 45 ff.; L. R. Palmer, "Aris invisa sedebat," *Mnemosyne* 6, 3rd Ser. (1938) 368–79; and, most recently, Büchner (above, note 14) 331 ff.; N. L. Hatch, "The Time Element in Interpretation of *Aeneid* 2. 575–76 and 585–87," *CP* 54 (1959) 255–57. Palmer views the sacrilege committed at altars as a *leitmotif* of Book II; his conclusion (379) ". . . not merely that the Helen episode is authentic, but that it constitutes the spiritual crisis of the second book" seems correct.

[58] See above, note 32.

3. Many editors suggest the transposition of lines or groups of lines. But if a passage is transferred from one series of proportions to another, several more or less perfect ratios may disappear. *We have here a useful control against editorial whim and wild conjecture.* E.g. in the catalogue of warriors in Book VII, Fowler suggests transposing six verses, 664–69, from Aventinus to Ufens, i.e. after 749.[59] Fowler's arguments have never impressed me as convincing, and certainly Aventinus, son of Hercules, should wear the lionskin of Hercules rather than Ufens. The Aventinus passage is a part of five proportions with ratios of .609, .622, .636, .618, .620, but with the removal of the six verses these same ratios become .529, .548, .579, .575, .568. If we add 664–69 to the Ufens passage, five other ratios change from .616, .615, .630, .615, .608 to .685, .653, .652, .634, .639. We have here additional arguments against Fowler's proposed transposition.

I have examined nine transpositions suggested by Ribbeck, Mackail, and others, with the following results. The removal of these passages from their contexts and their insertion elsewhere destroys 48 Golden Mean ratios and produces proportions ranging from .503 to .590 and from .650 to .780. Many examples of the Fibonacci series vanish, including five perfect .618 ratios. In other words, the mathematical structure of the poem gives no support to the transpositions proposed by modern editors but confirms the text of the poem as it has come down from antiquity.

4. I have in general followed the paragraphing in Hirtzel's Oxford edition, but I departed from it on occasion where the sense and the ratios indicated a different division; also, in many instances (especially in short passages) I made logical divisions where no paragraphs appeared. I now find that 187 of these new divisions, 99 between proportions and 88 between major and minor within proportions, are identical with the paragraphing in one or more of the following editions: Ladewig-Schaper-Deuticke-Jahn, Mackail, and Sabbadini.[60] This is an indication that these

[59] Fowler (above, note 52) 46 ff.; see also Mackail on 664–65.

[60] E.g. in Book III new paragraphs agreeing with my ratio divisions begin in the texts of Jahn (**J**), Mackail (**M**), or Sabbadini (**S**) after the following verses: 101 (M), 131 (M), 153 (M), 171 (JMS), 208 (JMS), 244 (JS), 257 (M), 267 (S), 283 (M), 319 (JS), 343 (JM), 380 (M), 395 (M), 409 (M), 432 (M), 440 (M), 460 (M), 612 (M), 668 (S), 706 (M). Hirtzel (**H**) paragraphs at none of the above places. For specific points connected with paragraphing, see above, notes 28, 30, 31, 33, and 36.

editions, and especially Mackail's, reproduce most faithfully the pattern of Vergil's thought, and they in turn support the proportions which are based on these same units of speech and narrative.

There are a number of places where editors disagree as to the beginning and the end of narrative units. Here the Golden Mean ratios may help to decide where the paragraphs are to be introduced. In Book II, e.g., the ratios favor new paragraphs after the lines listed in column 1 rather than those in column 2:

227 (J)	233 (S)
317 (JHS)	313 (M)
346 (S)	335 (JM)
436 (M)	437 (J)
623 (JHS)	620 (M)
670 (JHS)	672 (M)
795 (H)	794 (JMS)

The break between the end and the beginning of proportions indicates in many places where a paragraph might well be introduced into the text. Two important breaks occur after VIII.596,[61] and III.273; the latter concludes the second subdivision of the second main division of the book; 274 rather than 278 (JHS) marks the beginning of the Actium episode, and several proportions end with 273 or begin at 274. The ratios also favor paragraphs after the following: I.706 (not 708 with S); III.492 (not 491 with JM); V.853; VI.312, 449, 607, etc.

Mackail states that over-paragraphing which disturbs the continuity is to be avoided.[62] Most editors err in the opposite direction. A new edition of Vergil, based on narrative units as determined by the Golden Section, would combine the best features of the paragraphing in the Oxford text, Mackail's edition, and that of Sabbadini, and should produce the most readable text of the *Aeneid* to date.

5. Many suggestions made concerning Vergil's proposed revision of the poem appear in a new light when we consider the symmetrical arrangement of the proportions; e.g. Mackail thinks that Vergil contemplated discarding XII.593–611 and leaving Amata's end in silence.[63] I disagree. Amata's suicide is a necessary part of the tragedy of Turnus; his failure to face Aeneas

[61] See above, note 30.
[62] *The Aeneid*, lxiii.
[63] *The Aeneid*, 463; cf. 517 f.

leads her to believe him dead and she carries out her threat of XII.61–65.[64] Structurally also we should miss the episode; several Golden Mean ratios vanish, including that in the second subdivision of the second main section of the book, where the ratio changes from .621 to .718. An examination of other proposed revisions shows that they likewise disrupt the structure of the poem.

Book III has been called "the most incomplete and the least coherent in the whole Aeneid,"[65] and many believe that it would have undergone a thorough revision had Vergil lived longer; others have defended the book and have shown how the episodes were executed with meticulous care.[66] My own findings support the latter view. The close correlation of the narrative units and passages with ratios (as accurate as in I, IV, and VII, and more so than in V, VI, and XII) prove that Book III is constructed with far greater artistry than has usually been believed.

We shall never know what changes in wording and poetic expression Vergil would have made in his final revision, but the numerical symmetry of the *Aeneid* enables us to assert that he would *not* have done many of the things which have been suggested —such as the change of all half-lines to whole lines, the deletion of various passages, and the transposition of others. Thus, however fantastic the mathematical composition of the poem may seem, it favors a conservative treatment of the text.

There are two other aspects of the Golden Mean ratios which I touch upon very briefly.

1. The problem of the *Appendix Vergiliana*: the longer poems in the collection contain Golden Mean ratios, many of them in the Fibonacci series, and reveal also a close correlation between the proportions and the narrative units. This would mean Vergilian authorship if he were the only one who composed in this fashion. But when we find ratios corresponding to narrative units in Lucretius, Catullus, and Horace, we can not argue in favor of the authenticity of the so-called "minor poems."

[64] Cf. XII.598 f. See G. E. Duckworth, "Fate and Free Will in Vergil's *Aeneid*," *CJ* 51 (1955–56), 361 f., and note 32.

[65] Mackail, *The Aeneid*, 89; cf. Crump (above, note 57) 115: "Vergil had intended to give three years to the final revision of the Aeneid. . . . The most important task was the rewriting of III."

[66] See e.g. Lloyd (above, note 14) 133–51, and "*Aeneid* III and the Aeneas Legend," *AJP* 78 (1957) 382–400.

Certain results may be stated, however. The structure of the *Ciris* and the *Aetna*, each with four main divisions, resembles that of Lucretius I rather than that in the books of the *Aeneid*, and this might argue against the post-Vergilian authorship of the poems. The ratio in the *Dirae* as a whole includes 104–83, the lines considered by many to be a separate poem named *Lydia*, and thus gives support to those who maintain that the poem is a unity.[67]

The *Culex* has the strongest claim to Vergilian authorship, not only on the basis of external evidence, but also from the standpoint of composition; it has a tripartite structure similar to that in the books of the *Aeneid*, and the three main divisions (1–41, prooemium; 42–201, the shepherd, the serpent, and the gnat; 202–414, lament of the gnat; tomb and inscription) combine in a framework pattern (a + c = M = 254) to produce the Golden Section; 254/414 = .614. Also, the story itself as minor (158.6) is in proportion with the descriptive passages as major (255.4); 255.4/414 = .619.[68] This procedure is so similar to that in the *Georgics* as a whole where the technical and the descriptive sections are in proportion that the *Culex* seems less likely to be the work of a later imitator or forger than the work of a youthful Vergil who later used the same alternating technique in the composition of the *Georgics*.

2. The metrical patterns analyzed by Knight several years ago seem to go hand in hand with the smaller proportions. Knight's patterns are based upon heterodyne (clash of ictus and accent in the fourth foot) and homodyne (agreement of ictus and accent in the fourth foot).[69] I have examined many of these in passages where Golden Mean ratios appear, and the interesting result is this: in numerous proportions, the major has one type of metrical

[67] The ratio in the *Dirae* as a whole is derived from four alternating parts: (a) 1–47, (b) 48–81, (c) 82–103, and (d) 104–83, with b + d the major (114); 114/183 = .623. For the divisions after 47 and 81 (1–47, imprecation by fire; 48–81, imprecation by water), see C. Van der Graaf, *The Dirae, with Translation, Commentary and an Investigation of its Authorship* (Leiden 1945) 133. Van der Graaf argues against the view that the so-called *Lydia* is a separate poem.

[68] The actual narrative of the *Culex* is limited to 42–57, 98–122, 157–231a, and 372–414, a total of 158.6 verses.

[69] See W. F. J. Knight, *Accentual Symmetry in Vergil* (Oxford 1939); cf. his "Table of Fourth-Foot Texture" which follows p. 108: Catullus in 64 and Lucretius in Book I had used fourth-foot homodyne 61.25 and 51.49 per cent respectively; Vergil in his authentic works reduces the amount of fourth-foot homodyne to the following percentages: *Eclogues*, 37.27; *Georgics*, 33.45; *Aeneid*, 35.95. The *Culex* has 35.59 per cent, and in this respect also is Vergilian. (On the *Ciris*, see Knight, 42, 53.)

pattern, the minor another.[70] We have here triple correlation between narrative units, mathematical symmetry, and metrical structure—an added argument to support my view that Vergil's procedure throughout was deliberate.

To conclude, these various aspects of the mathematical structure of the *Aeneid* show conclusively, I believe, how important the Golden Section is both to the structure and to the content of the epic. Our increasing knowledge of Vergil's method of composition by narrative units coinciding with Golden Mean ratios may lead to a new and better text of the *Aeneid* and perhaps even indicate more clearly the nature of the poem as he would have left it, had he lived to give it the final revision.

[70] The following ratios will illustrate the correlation of narrative units, mathematical symmetry, and metrical patterns (="accentual symmetry"). I.124–141: 124–30, Neptune sees the effect of the storm (m=7), 131–41, Neptune rebukes the winds (M=11); 11/18=.611. The minor consists of a heterodyne sequence, the major of a framework and alternation (a=heterodyne, b=homodyne):

m=124–130; b a a a. a a a.

M=131–141: a a a b a. a b a b a b.

II.402–36: 402–23, temporary success of Trojans (M=22), 424–36, the Trojans slain (m=13); 22/35=.629. The major contains what Knight calls "one of the most elaborate symmetries of pure expanded alternation" (*Accentual Symmetry*, 66), the minor consists of simple alternations:

M=402–23: b a b b b. b a. a b b b. b a b; b b a a. b b a a.

m=424–36: a b a b a b b. b a b a a b.

The minor ends in a "released movement," i.e. two or more heterodynes followed by a homodyne (for the importance and the frequency of this pattern, see Knight, 48 ff.). The conclusion of the minor with a released movement in 434–36 (a a b) indicates that the punctuation with a full stop after 436, as in Mackail's edition, is correct; see above, 217.